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PLACE OF EXPERTS

Sample Work

Differential Equations: Applied Problems

Exponential decay, Newton's law of cooling, and mixing-tank models

This sample was prepared by an AceLocale subject-matter expert to illustrate the quality and structure of our done-for-you academic work.

(PART A)

Q:1 Let A = amount of lead

then $\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = KA$

This gives $A(t) = A_0 e^{Kt}$

Given that initially 1 gram of isotope is present, so $A_0 = 1$

So $A(t) = (1) e^{Kt} \Rightarrow A(t) = e^{Kt}$

Also given half lift is 3.3 hours

So $A(t) = \frac{1}{2} A_0$ $\quad t = 3.3$

$\hookrightarrow \frac{1}{2} A_0 = A_0 e^{K(3.3)}$

$\Rightarrow \frac{1}{2} = e^{3.3K}$

$\Rightarrow 3.3K = \ln\left(\frac{1}{2}\right) \Rightarrow K = \frac{\ln\left(\frac{1}{2}\right)}{3.3}$

OR $K = \frac{-\ln(2)}{3.3}$

So

$$A(t) = e^{\left(-\frac{\ln 2}{3.3}\right)t}$$

Now we need to find time it will take for 90% of lead to decay,

So after 90% decay, 10% will remain

So $A(t) = 10\% = 0.10$

So $0.1 = e^{\left(-\frac{\ln 2}{3.3}\right)t}$

$$\Rightarrow \ln(0.1) = \left(-\frac{\ln 2}{3.3}\right)t$$

OR $$t = \frac{3.3\ln(0.1)}{\ln(2)}$$

OR $$t \approx 10.96$$

So it will take 10.96 hours for 90% of lead to decay.

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Q:2 Let $T(t)$ is the temperature

So $\frac{dT}{dt} = K(T - T_i)$

Air temperature is $5°F$ So $T_i = 5$

So $\frac{dT}{dt} = K(T-5)$

Using Separation of variables

$$\int \frac{dT}{T-5} = \int K\,dt$$

$$\Rightarrow \ln|T-5| = Kt + C_1$$

$$\Rightarrow T-5 = e^{Kt+C_1}$$

$$\hookrightarrow T(t) = 5 + Ce^{Kt}$$

Given $T(1) = 55 \Rightarrow 55 = 5 + Ce^{K}$

$\Rightarrow 50 = Ce^{K} \Rightarrow C = 50e^{-K}$

Also $T(5) = 30 \Rightarrow 30 = 5 + Ce^{5K}$

$\Rightarrow 25 = 50e^{-K} \cdot e^{5K} \Rightarrow \frac{25}{50} = e^{4K}$

This gives $\ln\left(\frac{1}{2}\right) = 4K \Rightarrow K = -\frac{\ln 2}{4}$

So $C = 50e^{-\left(-\frac{\ln 2}{4}\right)} = 50e^{\ln(2^{1/4})} = 50(2)^{1/4}$

So $T(t) = 5 + 50(2)^{1/4} e^{\left(-\frac{\ln 2}{4}\right)t}$

Now $T(0) = 5 + 50(2)^{1/4} \cdot e^{0} \Rightarrow T(0) \approx 64.46$

So initial temperature was approximately $64.46°F$

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Q:3 There are two containers A & B.

Let $T(t)$ = temperature of bar.

Then $\frac{dT}{dt} = K(T - T_m)$

Find Find temperature of bare in container A.

initially $T_{m_A} = 0$ So $\frac{dT_A}{dt} = kT_A$

$$\Rightarrow \int \frac{dT_a}{T_A} = \int k\,dt \Rightarrow \ln|T_A| = kt + C$$

$$\Rightarrow T_A = ce^{kt}$$

Given $T_A(0) = 100 \Rightarrow 100 = ce^0 \Rightarrow C = 100$

So $T_A = 100e^{kt}$

And $T_A(1) = 90 \Rightarrow 90 = 100e^k \Rightarrow k = \ln\left(\frac{9}{10}\right)$

So Temperature of bar in container A is

$$T_A(t) = 100\,e^{(\ln \frac{9}{10})t}$$

After two minutes, in Container A,

$$T_A(2) = 100\,e^{(\ln\frac{9}{10})(2)} \approx 81$$

When This bar is taken to container B after two minutes, then in container B, initial temperature of bar is $T_B(0) = T_A(2) = 81$

Now find $T_B(t)$.

$$\frac{dT_B}{dt} = K(T_B - T_{B,M})$$

initially temperature of container B is $100^\circ C$

So $T_{B,M} = 100$

So $\frac{dT_B}{dt} = K(T_B - 100)$

Solving this DE $\Rightarrow \int \frac{dT_B}{T_B - 100} = \int K\, dt$

$\hookrightarrow \ln|T_B - 100| = Kt + C$

$\Rightarrow T_B(t) = 100 + Ce^{Kt}$

$T_B(0) = 81 \Rightarrow 81 = 100 + Ce^0 \Rightarrow C = -19$

So $T_B(t) = 100 - 19e^{Kt}$

Given $T_B(1) = 10^\circ + 81^\circ = 91$ As temperature of bar in container B after one minute rises to $10^\circ C$

Now, $T_B(1) = 100 - 19e^K = 91$

$\Rightarrow -9 = -19e^K \Rightarrow K = \ln\left(\frac{9}{19}\right)$

So $T_B(t) = 100 - 19e^{t \ln \frac{9}{19}}$

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Temperature of the bar will reach 99.9°C when

$$T_B(t) = 99.9$$

$$\Rightarrow \quad 100 - 19e^{t \ln \frac{9}{19}} = 99.9$$

$$\Rightarrow \quad 100 - 99.9 = 19e^{t \ln \frac{9}{19}}$$

$$\Rightarrow \quad 0.1 = 19e^{t \ln \frac{9}{19}}$$

$$\frac{0.1}{19} = e^{t \ln \frac{9}{19}}$$

$$\Rightarrow \quad t \ln\left(\frac{9}{19}\right) = \ln\left(\frac{0.1}{19}\right)$$

$$\Rightarrow \quad t = \frac{\ln\left(\frac{0.1}{19}\right)}{\ln\left(\frac{9}{19}\right)} \approx 7.022$$

So Bar is placed 2 minutes in container A and 7.022 minutes in container B.

So Total time taken by the bar to reach 99.9°C is $2 + 7.022 = 9.022$ minutes

Q:4 $A(t) =$ amount of salt is time t.

input rate $= \left(1 \frac{g}{L}\right)\left(4 \frac{L}{min}\right) = 4 \frac{g}{min}$

Output rate $= \left(4 \frac{L}{min}\right)\left(\frac{A(t)}{200}\right) = \frac{A(t)}{50}$

So $\frac{dA}{dt} = 4 - \frac{A}{50} \Rightarrow \frac{dA}{dt} = \frac{-1}{50}(A-200)$

initially 30 grams are dissolved

So $A(0) = 30$

Using Separation of variables

$$\int \frac{dA}{A-200} = \int -\frac{1}{50}\, dt$$

$$\ln|A-200| = -\frac{t}{50} + C_1$$

$$\Rightarrow A - 200 = e^{-\frac{t}{50}+C_1}$$

$$\Rightarrow A(t) = 200 + Ce^{-\frac{t}{50}}$$

As $A(0) = 30$

So $30 = 200 + Ce^{-0}$

$\hookrightarrow C = -170$

So

$$\boxed{A(t) = 200 - 170e^{-\frac{t}{50}}}$$

Q:5 $A(t)$ = amount of salt in t time.
Initially there is pure water So

$$A(0) = 0$$

$$\text{input rate} = \left(2 \frac{lb}{gal}\right)\left(5 \frac{gal}{min}\right) = 10 \frac{lb}{min}$$

$$\text{output rate} = \left(\frac{A}{500} \frac{lb}{gal}\right)\left(5 \frac{gal}{min}\right) = \frac{A}{100}$$

So $$\frac{dA}{dt} = 10 - \frac{A}{100} \Rightarrow A(t) = \frac{-1}{100}(A-100)$$

Using Separation of variables,

$$\int \frac{dA}{A-1000} = -\int \frac{1}{100} dt$$

$$\Rightarrow \ln|A-100| = -0.01t + C$$

This gives $A(t) = 100 + Ce^{-0.01t}$

Using $A(0) = 0 \Rightarrow 0 = 100 + Ce^{0} \Rightarrow C = -100$

$$\boxed{A(t) = 100 - 100e^{-0.01t}}$$

Q:6 we have $A(t) = 100 - 100\, e^{-0.01t}$

So $c(t) = \dfrac{A(t)}{500} = \dfrac{100 - 100\, e^{-0.01t}}{500}$

$\hookrightarrow$ $\boxed{c(t) = 0.2 - 0.2\, e^{-0.01t} \quad \frac{lb}{gal}}$

After 5 min

$C(5) = 0.2 - 0.2\, e^{-0.01(5)} \Rightarrow \boxed{C(5) = 0.00975 \ \frac{lb}{gal}}$

And $\lim\limits_{t\to\infty} (c(t)) = \lim\limits_{t\to\infty} \left(0.2 - 0.2\, e^{-0.01t}\right)$

$\hookrightarrow$ $\boxed{\lim\limits_{t\to\infty} (c(t)) = 0.2 \ \frac{lb}{gal}}$ $\leftarrow$ In long Run

Limiting values $= 0.2$ So half of limiting value $= 0.1$

Now $0.1 = 0.2 - 0.2\, e^{-0.01t}$

$\Rightarrow -0.1 = -0.2\, e^{-0.01t} \Rightarrow \frac{1}{2} = e^{-0.01t}$

$\hookrightarrow$ $\ln\left(\frac{1}{2}\right) = -0.01t$

$\hookrightarrow$ $t = \dfrac{\ln(1/2)}{-0.01}$

$\hookrightarrow$ $\boxed{t \approx 69.3 \ \text{min}}$