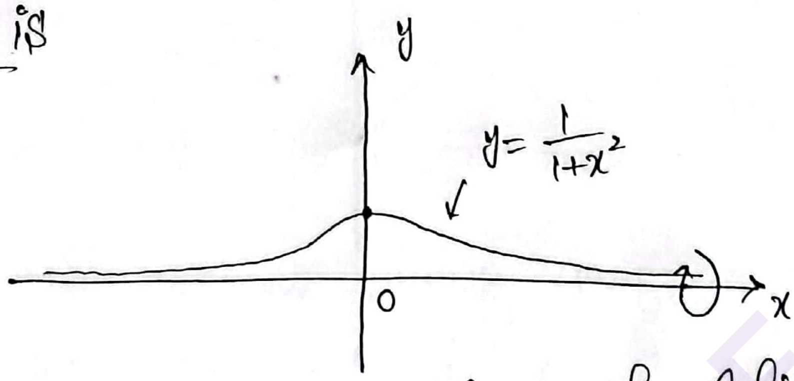
Q:4 Given $y = \frac{1}{1+x^2}$ and x-axis for $-\infty < x < \infty$

Graph is



Using disk Method, volume of Solid of revolution about x-axis is

$$V = \int_{-\infty}^{\infty} \pi\left(\left(\frac{1}{1+x^2}\right)^2 - (0)^2\right) dx$$

$$\hookrightarrow \quad V = \int_{-\infty}^{\infty} \frac{\pi}{(1+x^2)^2}\, dx \qquad \left(\text{Improper Integral}\right)$$

OR $$V = \lim_{R\to\infty}\left[\int_{-R}^{R} \frac{\pi}{(1+x^2)^2}\, dx\right]$$

Using Trigonometric Substitution

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

First solving indefinite integral.

NEXT PAGE

Q:4 Remaining

$$\int \frac{\pi}{(1+x^2)^2}\,dx = \int \frac{(\pi)(\sec^2\theta)}{(1+\tan^2\theta)^2}\,d\theta$$

$$= (\pi)\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta$$

$$= (\pi)\int \cos^2\theta\,d\theta$$

$$= (\pi)\int\left(\frac{1}{2}+\frac{\cos(2\theta)}{2}\right)d\theta$$

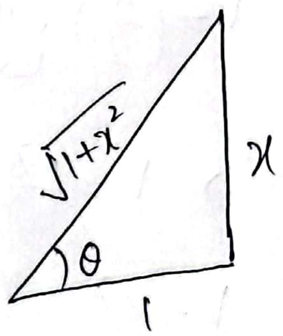
$$= \left(\frac{\pi}{2}\right)\int\left(1+\cos(2\theta)\right)d\theta$$

$$= \left(\frac{\pi}{2}\right)\left(\theta+\frac{\sin(2\theta)}{2}\right)+C$$

As $x = \tan\theta$

$\hookrightarrow \theta = \tan^{-1}(x)$

Using right angle triangle



So from triangle

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}$$

NEXT → PAGE

Q:4 Remaining

So
$$\int \frac{\pi}{(1+x^2)^2}dx = \left(\frac{\pi}{2}\right)\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + C$$
$$= \left(\frac{\pi}{2}\right)\left(\tan^{-1}(x) + \left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)\right) + C$$

$$\hookrightarrow \int \frac{\pi}{(1+x^2)^2}dx = \left(\frac{\pi}{2}\right)\left(\tan^{-1}(x) + \frac{x}{1+x^2}\right) + C$$

So Volume of required solid is

$$\text{Volume} = \lim_{R\to -\infty}\left[\frac{\pi}{2}\left(\tan^{-1}(x) + \frac{x}{1+x^2}\right)\right]_{-R}^{R}$$

$$V = \lim_{R\to\infty}\left(\left(\frac{\pi}{2}\right)\left[\left(\tan^{-1}(R) + \frac{R}{1+R^2}\right) - \left(\tan^{-1}(-R) + \frac{(-R)}{1+(-R)^2}\right)\right]\right)$$

$$V = \left(\frac{\pi}{2}\right)\lim_{R\to\infty}\left(\tan^{-1}(R) + \frac{R}{1+R^2} - \tan^{-1}(-R) + \frac{R}{1+R^2}\right)$$

$$V = \left(\frac{\pi}{2}\right)\lim_{R\to\infty}\left[\tan^{-1}(R) + \frac{2R}{1+R^2} - \tan^{-1}(-R)\right]$$

$$\hookrightarrow V = \left(\frac{\pi}{2}\right)\lim_{R\to\infty}\left(\tan^{-1}(R) + \frac{2}{R+\frac{1}{R}} - \tan^{-1}(-R)\right)$$

$$V = \left(\frac{\pi}{2}\right)\left(\tan^{-1}(\infty) + \frac{2}{\infty+0} - \tan^{-1}(-\infty)\right)$$

$$V = \left(\frac{\pi}{2}\right)\left(\frac{\pi}{2} + 0 - \left(-\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2}\right)\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\hookrightarrow V = \left(\frac{\pi}{2}\right)(\pi) \Rightarrow \boxed{\text{Volume} = \frac{\pi^2}{2}}\ \text{unit}^3$$